

Two-stage algorithms for covering array construction

Kaushik Sarkar and Charles J. Colbourn

School of Computing, Informatics, and Decision Systems Engineering
Arizona State University, PO Box 878809
Tempe, Arizona, 85287-8809, U.S.A.

June 23, 2016

Abstract

Modern software systems often consist of many different components, each with a number of options. Although unit tests may reveal faulty options for individual components, functionally correct components may interact in unforeseen ways to cause a fault. Covering arrays are used to test for interactions among components systematically. A two-stage framework, providing a number of concrete algorithms, is developed for the efficient construction of covering arrays. In the first stage, a time and memory efficient randomized algorithm covers most of the interactions. In the second stage, a more sophisticated search covers the remainder in relatively few tests. In this way, the storage limitations of the sophisticated search algorithms are avoided; hence the range of the number of components for which the algorithm can be applied is extended, without increasing the number of tests. Many of the framework instantiations can be tuned to optimize a memory-quality trade-off, so that fewer tests can be achieved using more memory. The algorithms developed outperform the currently best known methods when the number of components ranges from 20 to 60, the number of options for each ranges from 3 to 6, and t -way interactions are covered for $t \in \{5, 6\}$. In some cases a reduction in the number of tests by more than 50% is achieved.

Keywords: Covering array, Software interaction testing, Combinatorial construction algorithm

1 Introduction

Real world software and engineered systems are composed of many different components, each with a number of options, that are required to work together in a variety of circumstances. Components are factors, and options for a component form the levels of its factor. Although each level for an individual factor can be tested in isolation, faults in deployed software can arise from interactions among levels of different factors. When an interaction involves levels of t different factors, it is a t -way interaction. Testing for faults caused by t -way interactions for every t is generally infeasible, as a result of a combinatorial explosion. However, empirical research on real world software systems indicates that testing all possible 2-way (or 3-way) interactions would detect 70% (or 90%) of all faults [25]. Moreover, testing all possible 6-way interactions is sufficient for detection of 100% of all faults in the systems examined in [25]. Testing all possible t -way interactions for some $2 \leq t \leq 6$ is *pseudo-exhaustive testing* [24], and is accomplished with a combinatorial array known as a covering array.

Formally, let N , t , k , and v be integers with $k \geq t \geq 2$ and $v \geq 2$. A *covering array* $\text{CA}(N; t, k, v)$ is an $N \times k$ array A in which each entry is from a v -ary alphabet Σ , and for every $N \times t$ sub-array B of A and every $\mathbf{x} \in \Sigma^t$, there is a row of B that equals \mathbf{x} . Then t is the *strength* of the covering array, k is the number of *factors*, and v is the number of *levels*.

When k is a positive integer, $[k]$ denotes the set $\{1, \dots, k\}$. A t -way interaction is $\{(c_i, a_i) : 1 \leq i \leq t, c_i \in [k], c_i \neq c_j \text{ for } i \neq j, \text{ and } a_i \in \Sigma\}$. So an interaction is an assignment of levels from Σ to t of the k factors. $\mathcal{I}_{t,k,v}$ denotes the set of all $\binom{k}{t} v^t$ interactions for given t , k and v . An $N \times k$ array A *covers* the

interaction $\iota = \{(c_i, a_i) : 1 \leq i \leq t, c_i \in [k], c_i \neq c_j \text{ for } i \neq j, \text{ and } a_i \in \Sigma\}$ if there is a row r in A such that $A(r, c_i) = a_i$ for $1 \leq i \leq t$. When there is no such row in A , ι is *not covered* in A . Hence a $\text{CA}(N; t, k, v)$ covers all interactions in $\mathcal{I}_{t,k,v}$.

Covering arrays are used extensively for interaction testing in complex engineered systems. To ensure that all possible combinations of options of t components function together correctly, one needs examine all possible t -way interactions. When the number of components is k , and the number of different options available for each component is v , each row of $\text{CA}(N; t, k, v)$ represents a test case. The N test cases collectively test all t -way interactions. For this reason, covering arrays have been used in combinatorial interaction testing in varied fields like software and hardware engineering, design of composite materials, and biological networks [8, 24, 26, 32, 34].

The cost of testing is directly related to the number of test cases. Therefore, one is interested in covering arrays with the fewest rows. The smallest value of N for which $\text{CA}(N; t, k, v)$ exists is denoted by $\text{CAN}(t, k, v)$. Efforts to determine or bound $\text{CAN}(t, k, v)$ have been extensive; see [12, 14, 24, 31] for example. Naturally one would prefer to determine $\text{CAN}(t, k, v)$ exactly. Katona [22] and Kleitman and Spencer [23] independently showed that for $t = v = 2$, the minimum number of rows N in a $\text{CA}(N; 2, k, 2)$ is the smallest N for which $k \leq \binom{N-1}{\lfloor \frac{N}{2} \rfloor}$. Exact determination of $\text{CAN}(t, k, v)$ for other values of t and v has remained open. However, some progress has been made in determining upper bounds for $\text{CAN}(t, k, v)$ in the general case; for recent results, see [33].

For practical applications such bounds are often unhelpful, because one needs explicit covering arrays to use as test suites. Explicit constructions can be recursive, producing larger covering arrays using smaller ones as ingredients (see [14] for a survey), or direct. Direct methods for some specific cases arise from algebraic, geometric, or number-theoretic techniques; general direct methods are computational in nature. Indeed when k is relatively small, the best known results arise from computational techniques [13], and these are in turn essential for the successes of recursive methods. Unfortunately, the existing computational methods encounter difficulties as k increases, but is still within the range needed for practical applications. Typically such difficulties arise either as a result of storage or time limitations or by producing covering arrays that are too big to compete with those arising from simpler recursive methods.

Cohen [11] discusses commercial software where the number of factors often exceeds 50. Aldaco et al. [1] analyze a complex engineered system having 75 factors, using a variant of covering arrays. Android [3] uses a *Configuration* class to describe the device configuration; there are 17 different configuration parameters with 3 – 20 different levels. In each of these cases, while existing techniques are effective when the strength is small, these moderately large values of k pose concerns for larger strengths.

In this paper, we focus on situations in which every factor has the same number of levels. These cases have been most extensively studied, and hence provide a basis for making comparisons. In practice, however, often different components have different number of levels, which is captured by extending the notion of a covering array. A *mixed covering array* $\text{MCA}(N; t, k, (v_1, v_2, \dots, v_k))$ is an $N \times k$ array in which the i th column contains v_i symbols for $1 \leq i \leq k$. When $\{i_1, \dots, i_t\} \subseteq \{1, \dots, k\}$ is a set of t columns, in the $N \times t$ subarray obtained by selecting columns i_1, \dots, i_t of the MCA, each of the $\prod_{j=1}^t v_{i_j}$ distinct t -tuples appears as a row at least once. Although we examine the uniform case in which $v_1 = \dots = v_k$, the methods developed here can all be directly applied to mixed covering arrays as well.

Inevitably, when $k > \max(t + 1, v + 2)$, a covering array must cover some interactions more than once, for if not they are orthogonal arrays [20]. Treating the rows of a covering array in a fixed order, each row covers some number of interactions not covered by any earlier row. For a variety of known constructions, the initial rows cover many new interactions, while the later ones cover very few [7]. Comparing this rate of coverage for a purely random method and for one of the sophisticated search techniques, one finds little difference in the initial rows, but very substantial differences in the final ones. This suggests strategies to build the covering array in stages, investing more effort as the number of remaining uncovered interactions declines.

In this paper we propose a new algorithmic framework for covering array construction, the *two-stage framework*. In the first stage, a randomized row construction method builds a specified number of rows to cover all but at most a specified, small number of interactions. As we see later, by dint of being randomized

this uses very little memory. The second stage is a more sophisticated search that adds few rows to cover the remaining uncovered interactions. We choose search algorithms whose requirements depend on the number of interactions to be covered, to profit from the fact that few interactions remain. By mixing randomized and deterministic methods, we hope to retain the fast execution and small storage of the randomized methods, along with the accuracy of the deterministic search techniques.

We introduce a number of algorithms within the two-stage framework. Some improve upon best known bounds on $\text{CAN}(t, k, v)$ (see [33]) in principle. But our focus is on the practical consequences: The two-stage algorithms are indeed quite efficient for higher strength ($t \in \{5, 6\}$) and moderate number of levels ($v \in \{3, 4, 5, 6\}$), when the number of factors k is moderately high (approximately in the range of 20 – 80 depending on value of t and v). In fact, for many combination of t, k and v values the two-stage algorithms beat the previously best known bounds.

Torres-Jimenez et al. [36] explore a related two-stage strategy. In their first stage, an in-parameter-order greedy strategy (as used in ACTS [24]) adds a column to an existing array; in their second stage, simulated annealing is applied to cover the remaining interactions. They apply their methods when $t = v = 3$, when the storage and time requirements for both stages remain acceptable. In addition to the issues in handling larger strengths, their methods provide no *a priori* bound on the size of the resulting array. In contrast with their methods, ours provide a guarantee prior to execution with much more modest storage and time.

The rest of the paper is organized as follows. Section 2 reviews algorithmic methods of covering array construction, specifically the randomized algorithm and the density algorithm. This section contrasts these two methods and points out their limitations. Then it gives an intuitive answer to the question of why a two stage based strategy might work and introduces the general two-stage framework. Section 3 introduces some specific two-stage algorithms. Section 3.1 analyzes and evaluates the naïve strategy. Section 3.2 describes a two-stage algorithm that combines the randomized and the density algorithm. Section 3.3 introduces graph coloring based techniques in the second stage. Section 3.4 examines the effect of group action on the size of the constructed covering arrays. Section 4 compares the results of various two-stage algorithms with the presently best known sizes. In Section 5 we discuss the Lovász local lemma (LLL) bounds on $\text{CAN}(t, k, v)$ and provide a Moser-Tardos type randomized algorithm for covering array construction that matches the bound. Although the bound was known [18], the proof was non-constructive, and a constructive algorithm to match this bound seems to be absent in the literature. We explore potentially better randomized algorithms for the first stage using LLL based techniques. We also obtain a two-stage bound that improves the LLL bound for $\text{CAN}(t, k, v)$. We conclude the paper in Section 6.

2 Algorithmic construction of covering arrays

Available algorithms for the construction of covering arrays are primarily heuristic in nature; indeed exact algorithms have succeeded for very few cases. Computationally intensive metaheuristic search methods such as simulated annealing, tabu search, constraint programming, and genetic algorithms have been employed when the strength is relatively small or the number of factors and levels is small. These methods have established many of the best known bounds on sizes of covering arrays [13], but for many problems of practical size their time and storage requirements are prohibitive. For larger problems, the best available methods are greedy. The IPO family of algorithms [24] repeatedly adds one column at a time, and then adds new rows to ensure complete coverage. In this way, at any point in time, the status of $v^t \binom{k-1}{t-1}$ interactions may be stored. AETG [10] pioneered a different method, which greedily selects one row at a time to cover a large number of as-yet-uncovered interactions. They establish that if a row can be chosen that covers the maximum number, a good *a priori* bound on the size of the covering array can be computed. Unfortunately selecting the maximum is NP-hard, and even if one selects the maximum there is no guarantee that the covering array is the smallest possible [7], so AETG resorts to a good heuristic selection of the next row by examining the stored status of $v^t \binom{k}{t}$ interactions. None of the methods so far mentioned therefore guarantee to reach an *a priori* bound. An extension of the AETG strategy, the density algorithm [5, 6, 15], stores additional statistics for all $v^t \binom{k}{t}$ interactions in order to ensure the selection of a good next row, and hence guarantees to produce an array with at most the precomputed number of rows. Variants of the density

Algorithm 1: A randomized algorithm for covering array construction.

Input: t : strength of the covering array, k : number of factors, v : number of levels for each factor

Output: A : a $\text{CA}(N; t, k, v)$

```

1 Set  $N := \left\lceil \frac{\log \binom{k}{t} + t \log v}{\log \left( \frac{v^t}{v^t - 1} \right)} \right\rceil$ ;
2 repeat
3   Construct an  $N \times k$  array  $A$  where each entry is chosen independently and uniformly at random
   from a  $v$ -ary alphabet;
4   Set  $\text{covered} := \text{true}$ ;
5   for each interaction  $\iota \in \mathcal{I}_{t,k,v}$  do
6     if  $\iota$  is not covered in  $A$  then
7       Set  $\text{covered} := \text{false}$ ;
8       break;
9   end
10  end
11 until  $\text{covered} = \text{true}$ ;
12 Output  $A$ ;
```

algorithm have proved to be most effective for problems of moderately large size. For even larger problems, pure random approaches have been applied.

To produce methods that provide a guarantee on size, it is natural to focus on the density algorithm in order to understand its strengths and weaknesses. To do this, we contrast it with a basic randomized algorithm. Algorithm 1 shows a simple randomized algorithm for covering array construction. The algorithm constructs an array of a particular size randomly and checks whether all the interactions are covered. It repeats until it finds an array that covers all the interactions.

A $\text{CA}(N; t, k, v)$ with $N = \frac{\log \binom{k}{t} + t \log v}{\log \left(\frac{v^t}{v^t - 1} \right)}$ is guaranteed to exist:

Theorem 1. [21, 27, 35] (*Stein-Lovász-Johnson (SLJ) bound*): Let t, k, v be integers with $k \geq t \geq 2$, and $v \geq 2$. Then as $k \rightarrow \infty$,

$$\text{CAN}(t, k, v) \leq \frac{\log \binom{k}{t} + t \log v}{\log \left(\frac{v^t}{v^t - 1} \right)}$$

In fact, the probability that the $N \times k$ array constructed in line 3 of Algorithm 1 is a valid covering array is high enough that the expected number of times the loop in line 2 is repeated is a small constant.

An alternative strategy is to add rows one by one instead of constructing the full array at the outset. We start with an empty array, and whenever we add a new row we ensure that it covers at least the expected number of previously uncovered interactions for a randomly chosen row. The probability that an uncovered interaction is covered by a random row is $1/v^t$. If the number of uncovered interactions is u , then by linearity of expectation, the expected number of newly covered interactions in a randomly chosen row is uv^{-t} . If each row added covers exactly this expected number, we obtain the same number of rows as the SLJ bound, realized in Algorithm 1. But because the actual number of newly covered interactions is always an integer, each added row covers at least $\lceil uv^{-t} \rceil$ interactions. This is especially helpful towards the end when the expected number is a small fraction.

Algorithm 2 follows this strategy. Again the probability that a randomly chosen row covers at least the expected number of previously uncovered interactions is high enough that the expected number of times the row selection loop in line 6 of Algorithm 2 is repeated is bounded by a small constant.

We can obtain an upper bound on the size produced by Algorithm 2 by assuming that each new row added covers exactly $\lceil uv^{-t} \rceil$ previously uncovered interactions. This bound is the *discrete Stein-Lovász-*

Algorithm 2: A randomized algorithm for covering array construction using the discrete SLJ strategy.

Input: t : strength of the covering array, k : number of factors, v : number of levels for each factor
Output: A : a $\text{CA}(N; t, k, v)$

- 1 Let A be an empty array;
- 2 Initialize a table T indexed by all $\binom{k}{t}v^t$ interactions, marking every interaction “uncovered”;
- 3 **while** there is an interaction marked “uncovered” in T **do**
- 4 Let u be the number of interactions marked “uncovered” in T ;
- 5 Set $\text{expectedCoverage} := \lceil \frac{u}{v^t} \rceil$;
- 6 **repeat**
- 7 Let r be a row of length k where each entry is chosen independently and uniformly at random from a v -ary alphabet;
- 8 Let coverage be the number of “uncovered” interactions in T that are covered in row r ;
- 9 **until** $\text{coverage} > \text{expectedCoverage}$;
- 10 Add r to A ;
- 11 Mark all interactions covered by r as “covered” in T ;
- 12 **end**
- 13 Output A ;

Johnson (discrete SLJ) bound. Figure 1 compares the sizes of covering arrays obtained from the SLJ and the discrete SLJ bounds for different values of k when $t = 6$ and $v = 3$. Consider a concrete example, when $t = 5$, $k = 20$, and $v = 3$. The SLJ bound guarantees the existence of a covering array with 12499 rows, whereas the discrete SLJ bound guarantees the existence of a covering array with only 8117 rows.

The density algorithm replaces the loop at line 6 of Algorithm 2 by a conditional expectation derandomized method. For fixed v and t the density algorithm selects a row efficiently (time polynomial in k) and deterministically that is guaranteed to cover at least $\lceil uv^{-t} \rceil$ previously uncovered interactions. In practice, for small values of k the density algorithm works quite well, often covering many more interactions than the minimum. Many of the currently best known $\text{CAN}(t, k, v)$ upper bounds are obtained by the density algorithm in combination with various post-optimization techniques [13].

However, the practical applicability of Algorithm 2 and the density algorithm is limited by the storage of the table T , representing each of the $\binom{k}{t}v^t$ interactions. Even when $t = 6$, $v = 3$, and $k = 54$, there are 18,828,003,285 6-way interactions. This huge memory requirement renders the density algorithm impractical for rather small values of k when $t \in \{5, 6\}$ and $v \geq 3$. We present an idea to circumvent this large requirement for memory, and develop it in full in Section 3.

2.1 Why does a two stage based strategy make sense?

Compare the two extremes, the density algorithm and Algorithm 1. On one hand, Algorithm 1 does not suffer from any substantial storage restriction, but appears to generate many more rows than the density algorithm. On the other hand, the density algorithm constructs fewer rows for small values of k , but becomes impractical when k is moderately large. One wants algorithms that behave like Algorithm 1 in terms of memory, but yield a number of rows competitive with the density algorithm.

For $t = 6$, $k = 16$, and $v = 3$, Figure 2 compares the coverage profile for the density algorithm and Algorithm 1. We plot the number of newly covered interactions for each row in the density algorithm, and the expected number of newly covered interactions for each row for Algorithm 1. The qualitative features exhibited by this plot are representative of the rates of coverage for other parameters.

Two key observations are suggested by Figure 2. First, the expected coverage in the initial random rows is similar to the rows chosen by the density algorithm. In this example, the partial arrays consisting of the first 1000 rows exhibit similar coverage, yet the randomized algorithm needed no extensive bookkeeping. Secondly, as later rows are added, the judicious selections of the density algorithm produce much larger coverage per row than Algorithm 1. Consequently it appears sensible to invest few computational resources

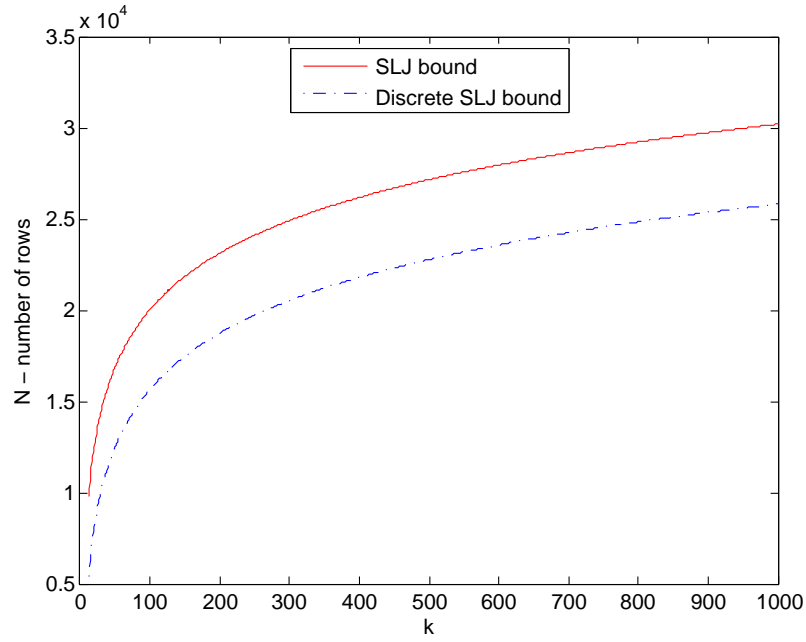


Figure 1: Comparison of covering array sizes obtained from SLJ bound and discrete SLJ bound for different values of k , when $t = 6$ and $v = 3$.

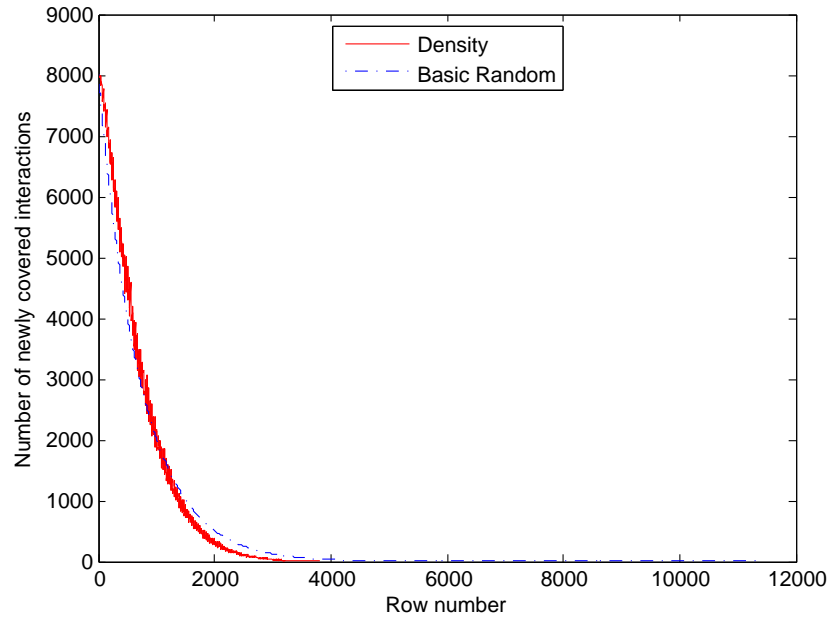


Figure 2: For $t = 6$, $k = 16$ and $v = 3$, the actual number of newly covered interactions of the density algorithm and the expected number of newly covered interactions in a random array.

on the initial rows, while making more careful selections in the later ones. This forms the blueprint of our general *two-stage* algorithmic framework shown in Algorithm 3.

Algorithm 3: The general two-stage framework for covering array construction.

Input: t : strength of the required covering array, k : number of factors, v : number of levels for each factor
Output: A : a $\text{CA}(N; t, k, v)$

- 1 Choose a number n of rows and a number ρ of interactions;
// First Stage
- 2 Use a randomized algorithm to construct an $n \times k$ array A' ;
- 3 Ensure that A' covers all but at most ρ interactions;
- 4 Make a list L of interactions that are not covered in A' (L contains at most ρ interactions);
// Second Stage
- 5 Use a deterministic procedure to add $N - n$ rows to A' to cover all the interactions in L ;
- 6 Output A ;

A specific covering array construction algorithm results by specifying the randomized method in the first stage, the deterministic method in the second stage, and the computation of n and ρ . Any such algorithm produces a covering array, but we wish to make selections so that the resulting algorithms are practical while still providing a guarantee on the size of the array. In Section 3 we describe different algorithms from the two-stage family, determine the size of the partial array to be constructed in the first stage, and establish upper bound guarantees. In Section 4 we explore how good the algorithms are in practice.

3 Two-stage framework

For the first stage we consider two methods:

| | |
|------|---------------------------------|
| Rand | the basic randomized algorithm |
| MT | the Moser-Tardos type algorithm |

We defer the development of method MT until Section 5. Method Rand uses a simple variant of Algorithm 1, choosing a random $n \times k$ array.

For the second stage we consider four methods:

| | |
|--------|---|
| Naive | the naïve strategy, one row per uncovered interaction |
| Greedy | the online greedy coloring strategy |
| Den | the density algorithm |
| Col | the graph coloring algorithm |

Using these abbreviations, we adopt a uniform naming convention for the algorithms: $\text{TS}\langle A, B \rangle$ is the algorithm in which A is used in the first stage, and B is used in the second stage. For example, $\text{TS}\langle \text{MT}, \text{Greedy} \rangle$ denotes a two-stage algorithm where the first stage is a Moser-Tardos type randomized algorithm and the second stage is a greedy coloring algorithm. Later when the need arises we refine these algorithm names.

3.1 One row per uncovered interaction in the second stage ($\text{TS}\langle \text{Rand}, \text{Naive} \rangle$)

In the second stage each of the uncovered interactions after the first stage is covered using a new row. Algorithm 4 describes the method in more detail.

This simple strategy improves on the basic randomized strategy when n is chosen judiciously. For example, when $t = 6$, $k = 54$ and $v = 3$, Algorithm 1 constructs a covering array with 17,236 rows. Figure 3 plots an upper bound on the size of the covering array against the number n of rows in the partial array.

Algorithm 4: Naïve two-stage algorithm (TS $\langle \text{Rand}, \text{Naive} \rangle$).

Input: t : strength of the covering array, k : number of factors, v : number of levels for each factor

Output: A : a $\text{CA}(N; t, k, v)$

```

1 Let  $n := \frac{\log \binom{k}{t} + t \log v + \log \log \left( \frac{v^t}{v^t - 1} \right)}{\log \left( \frac{v^t}{v^t - 1} \right)}$ ;
2 Let  $\rho = \frac{1}{\log \left( \frac{v^t}{v^t - 1} \right)}$ ;
3 repeat
4   Let  $A$  be an  $n \times k$  array where each entry is chosen independently and uniformly at random from
   a  $v$ -ary alphabet;
5   Let  $\text{uncovNum} := 0$  and  $\text{unCovList}$  be an empty list of interactions;
6   Set  $\text{covered} := \text{true}$ ;
7   for each interaction  $\iota \in \mathcal{I}_{t,k,v}$  do
8     if  $\iota$  is not covered in  $A$  then
9       Set  $\text{uncovNum} := \text{uncovNum} + 1$ ;
10      Add  $\iota$  to  $\text{unCovList}$ ;
11      if  $\text{uncovNum} > \rho$  then
12        Set  $\text{covered} := \text{false}$ ;
13        break;
14      end
15    end
16  end
17 until  $\text{covered} = \text{true}$ ;
18 for each interaction  $\iota \in \text{unCovList}$  do
19   Add a row to  $A$  that covers  $\iota$ ;
20 end
21 Output  $A$ ;
```

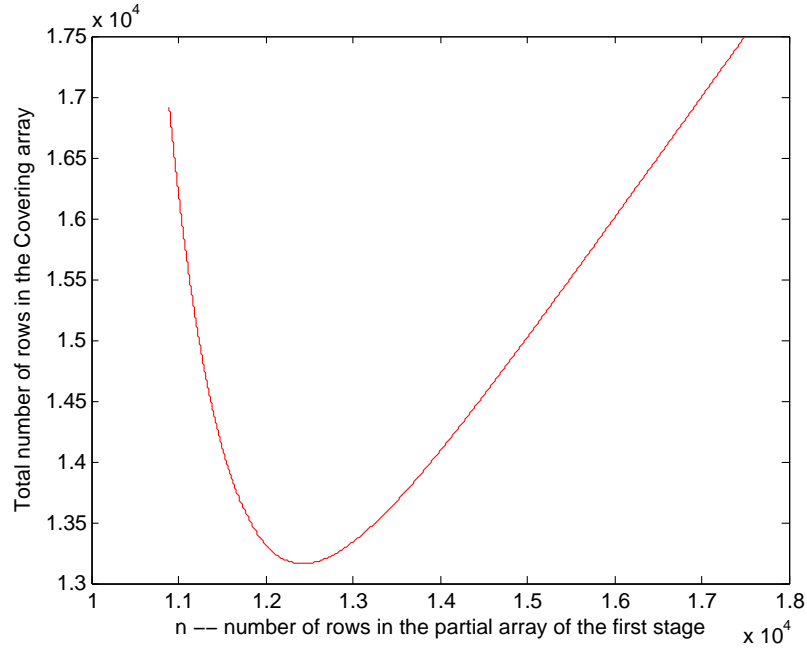


Figure 3: An upper bound on the size of the covering array against n , the size of the partial array constructed in the first stage when $t = 6$, $k = 54$, and $v = 3$, with one new row added per uncovered interaction in the second stage. The minimum size of 13,162 is obtained when $n = 12,402$. Algorithm 1 requires 17,236 rows, and the currently best known covering array has 17,197 rows.

The smallest covering array is obtained when $n = 12,402$ which, when completed, yields a covering array with at most 13,162 rows—a big improvement over Algorithm 1.

A theorem from [33] tells us the optimal value of n in general:

Theorem 2. [33] *Let t, k, v be integers with $k \geq t \geq 2$, and $v \geq 2$. Then*

$$\text{CAN}(t, k, v) \leq \frac{\log \binom{k}{t} + t \log v + \log \log \left(\frac{v^t}{v^t - 1} \right) + 1}{\log \left(\frac{v^t}{v^t - 1} \right)}.$$

The bound is obtained by setting $n = \frac{\log \binom{k}{t} + t \log v + \log \log \left(\frac{v^t}{v^t - 1} \right)}{\log \left(\frac{v^t}{v^t - 1} \right)}$. The expected number of uncovered interactions is exactly $\rho = 1 / \log \left(\frac{v^t}{v^t - 1} \right)$.

Figure 4 compares SLJ, discrete SLJ and two-stage bounds for $k \leq 100$, when $t = 6$ and $v = 3$. The two-stage bound does not deteriorate in comparison to discrete SLJ bound as k increases; it consistently takes only 307-309 more rows. Thus when $k = 12$ the two-stage bound requires only 6% more rows and when $k = 100$ only 2% more rows than the discrete SLJ bound.

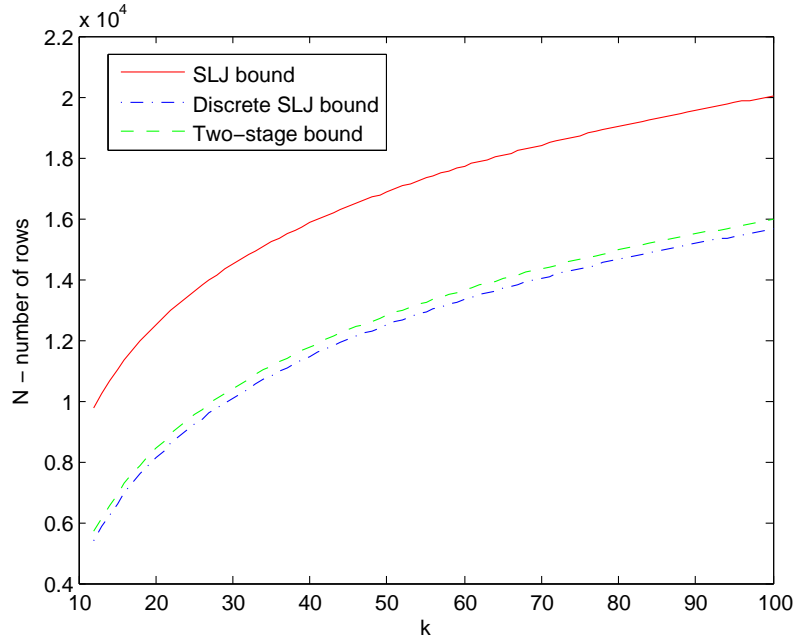


Figure 4: Comparison of covering array sizes obtained from SLJ bound, discrete SLJ bound and two-stage bound for $k \leq 100$, when $t = 6$ and $v = 3$. In this range of k values the two-stage bound requires 307-309 more rows than the discrete SLJ bound, that is, 2-6% more rows.

To ensure that the loop in line 7 of Algorithm 4 does not repeat too many times we need to know the probability with which a random $n \times k$ array leaves at most ρ interactions uncovered. Using Chebyshev's inequality and the second moment method developed in [2, Chapter 4], we next show that in a random $n \times k$ array the number of uncovered interactions is almost always close to its expectation, i.e. $\binom{k}{t} v^t \left(1 - \frac{1}{v^t}\right)^n$. Substituting the value of n from line 1, this expected value is equal to μ , as in line 2. Therefore, the probability that a random $n \times k$ array covers the desired number of interactions is constant, and the expected number of times the loop in line 7 is repeated is also a constant (around 2 in practice).

Because the theory of the second moment method is developed in considerable detail in [2], here we briefly mention the relevant concepts and results. Suppose that $X = \sum_{i=1}^m X_i$, where X_i is the indicator random variable for event A_i for $1 \leq i \leq m$. For indices i, j , we write $i \sim j$ if $i \neq j$ and the events A_i, A_j are not independent. Also suppose that X_1, \dots, X_m are *symmetric*, i.e. for every $i \neq j$ there is a measure preserving mapping of the underlying probability space that sends event A_i to event A_j . Define $\Delta^* = \sum_{j \sim i} \Pr[A_j | A_i]$. Then by [2, Corollary 4.3.4]:

Lemma 3. [2] *If $E[X] \rightarrow \infty$ and $\Delta^* = o(E[X])$ then $X \sim E[X]$ almost always.*

In our case, A_i denotes the event that the i th interaction is *not* covered in a $n \times k$ array where each entry is chosen independently and uniformly at random from a v -ary alphabet. Then $\Pr[X_i] = (1 - \frac{1}{v^t})^n$. Because there are $\binom{k}{t} v^t$ interactions in total, by linearity of expectation, $E[X] = \binom{k}{t} v^t (1 - \frac{1}{v^t})^n$, and $E[X] \rightarrow \infty$ as $k \rightarrow \infty$.

Distinct events A_i and A_j are independent if the i th and j th interactions share no column. Therefore, the event A_i is not independent of at most $t \binom{k}{t-1}$ other events A_j . So $\Delta^* = \sum_{j \sim i} \Pr[A_j | A_i] \leq \sum_{j \sim i} 1 \leq t \binom{k}{t-1} = o(E[X])$ when v and t are constants. By Lemma 3, the number of uncovered interactions in a random $n \times k$ array is close to the expected number of uncovered interactions. This guarantees that Algorithm 4 is an efficient randomized algorithm for constructing covering arrays with a number of rows upper bounded by Theorem 2.

In keeping with the general two-stage framework, Algorithm 4 does not store the coverage status of each interaction. We only need store the interactions that are uncovered in A , of which there are at most $\rho = \frac{1}{\log(\frac{v^t}{v^t-1})} \approx v^t$. This quantity depends only on v and t and is independent of k , so is effectively a constant that is much smaller than $\binom{k}{t} v^t$, the storage requirement for the density algorithm. Hence the algorithm can be applied to a higher range of k values.

Although Theorem 5 provides asymptotically tighter bounds than Theorem 2, in a range of k values that are relevant for practical application, Theorem 2 provides better results. Figure 5 compares the bounds on $\text{CAN}(t, k, v)$ with the currently best known results.

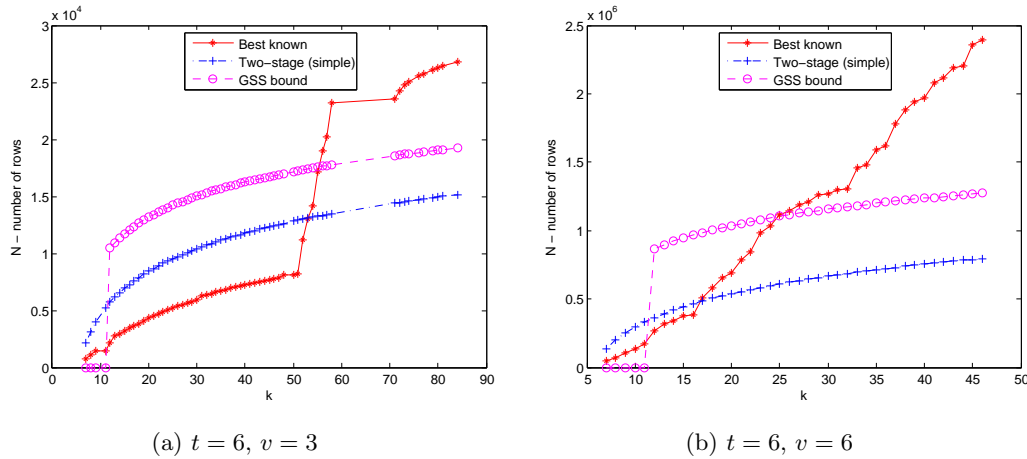


Figure 5: Comparison of GSS bound and two-stage bound with the currently best known results

3.2 The density algorithm in the second stage (TS (Rand, Den))

Next we apply the density algorithm in the second stage. Figure 6 plots an upper bound on the size of the covering array against the size of the partial array constructed in the first stage when the density algorithm is used in the second stage, and compares it with TS (Rand, Naive). The size of the covering array decreases

as n decreases. This is expected because with smaller partial arrays, more interactions remain for the second stage to be covered by the density algorithm. In fact if we cover all the interactions using the density algorithm (as when $n = 0$) we would get an even smaller covering array. However, our motivation was precisely to avoid doing that. Therefore, we need a "cut-off" for the first stage.

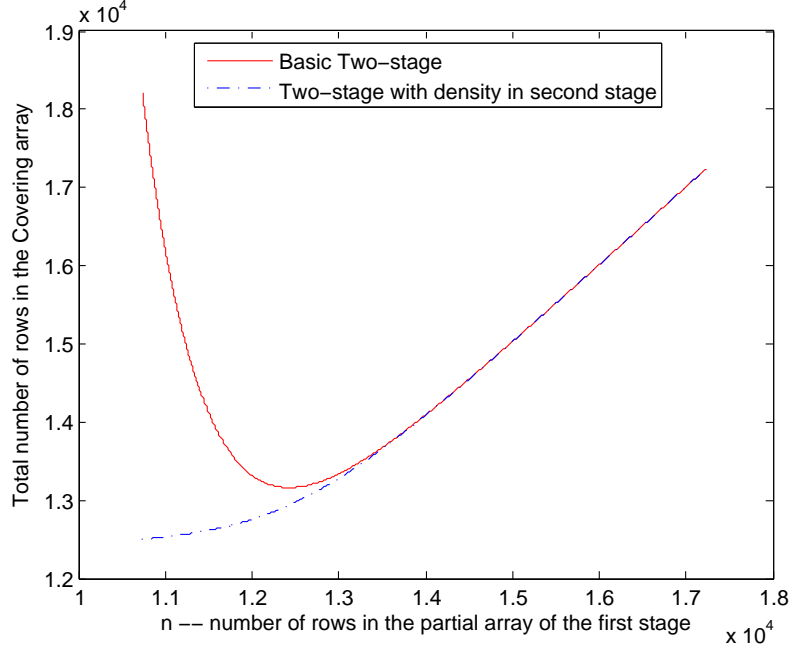


Figure 6: Comparison of covering array sizes from two-stage algorithms with Den and Naive in the second stage. With Den there is no minimum point in the curve; the size of the covering array keeps decreasing as we leave more uncovered interactions for the second stage.

We are presented with a trade-off. If we construct a smaller partial array in the first stage, we obtain a smaller covering array overall. But we then pay for more storage and computation time for the second stage. To appreciate the nature of this trade-off, look at Figure 7, which plots an upper bound on the covering array size and the number of uncovered interactions in the first stage against n . The improvement in the covering array size plateaus after a certain point. The three horizontal lines indicate ρ ($\approx v^t$), 2ρ and 3ρ uncovered interactions in the first stage. (In the naïve method of Section 3.1, the partial array after the first stage leaves at most ρ uncovered interactions.) In Figure 7 the final covering array size appears to plateau when the number of uncovered interactions left by the first stage is around 2ρ . After that we see diminishing returns — the density algorithm needs to cover more interactions in return for a smaller improvement in the covering array size.

Let r be the maximum number of interactions allowed to remain uncovered after the first stage. Then r can be specified in the two-stage algorithm. To accommodate this, we denote by $\text{TS}\langle A, B; r \rangle$ the two-stage algorithm where A is the first stage strategy, B is the second stage strategy, and r is the maximum number of uncovered interactions after the first stage. For example, $\text{TS}\langle \text{Rand}, \text{Den}; 2\rho \rangle$ applies the basic randomized algorithm in the first stage to cover all but at most 2ρ interactions, and the density algorithm to cover the remaining interactions in the second stage.

3.3 Coloring in the second stage ($\text{TS}\langle \text{Rand}, \text{Col} \rangle$ and $\text{TS}\langle \text{Rand}, \text{Greedy} \rangle$)

Now we describe strategies using graph coloring in the second stage. Construct a graph $G = (V, E)$, the *incompatibility graph*, in which V is the set of uncovered interactions, and there is an edge between two

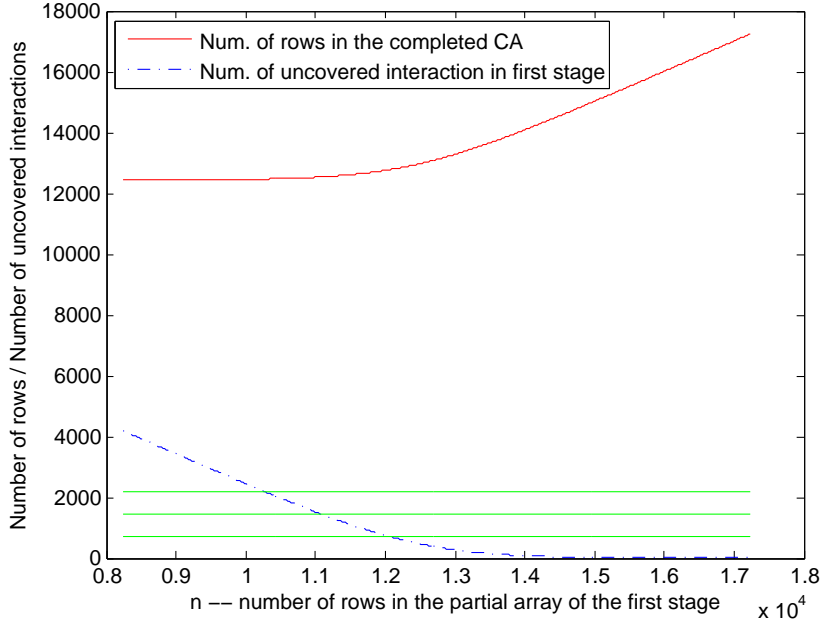


Figure 7: Final covering array size against the number of uncovered interactions after the first stage. As the size n of the partial array decreases, the number of uncovered interactions in the first stage increases. Den is used in the second stage. From bottom to top, the green lines denote ρ , 2ρ , and 3ρ uncovered interactions.

interactions exactly when they share a column in which they have different symbols. A single row can cover a set of interactions if and only if it forms an independent set in G . Hence the minimum number of rows required to cover all interactions of G is exactly its chromatic number $\chi(G)$, the minimum number of colors in a proper coloring of G . Graph coloring is an NP-hard problem, so we employ heuristics to bound the chromatic number. Moreover, G only has vertices for the *uncovered* interactions after the first stage, so its size is small relative to the total number of interactions.

The expected number of edges in the incompatibility graph after choosing n rows uniformly at random is $\gamma = \left(\frac{1}{2}\right) \binom{k}{t} v^t \sum_{i=1}^t \binom{t}{i} \binom{k-t}{t-i} (v^t - v^{t-i}) \left(1 - \frac{1}{v^t}\right)^n \left(1 - \frac{1}{(v^t - v^{t-i})}\right)^n$. Using the elementary upper bound on the chromatic number $\chi \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$, where m is the number of edges [16, Chapter 5.2], we can surely cover the remaining interactions with at most $\frac{1}{2} + \sqrt{2m + \frac{1}{4}}$ rows.

The actual number of edges m that remain after the first stage is a random variable with mean γ . In principle, the first stage could be repeatedly applied until $m \leq \gamma$, so we call $m = \gamma$ the *optimistic estimate*. To ensure that the first stage is expected to be run a small constant number of times, we increase the estimate. With probability more than $1/2$ the incompatibility graph has $m \leq 2\gamma$ edges, so $m = 2\gamma$ is the *conservative estimate*.

For $t = 6$, $k = 56$, and $v = 3$, Figure 8 shows the effect on the minimum number of rows when the bound on the chromatic number in the second stage is used, for the conservative or optimistic estimates. The Naïve method is plotted for comparison. Better coloring bounds shift the minima leftward, reducing the number of rows produced in both stages.

Thus far we have considered bounds on the chromatic number. Better estimation of $\chi(G)$ is complicated by the fact that we do not have much information about the structure of G until the first stage is run. In practice, however, G is known after the first stage and hence an algorithmic method to bound its chromatic number can be applied. Because the number of vertices in G equals the number of uncovered interactions

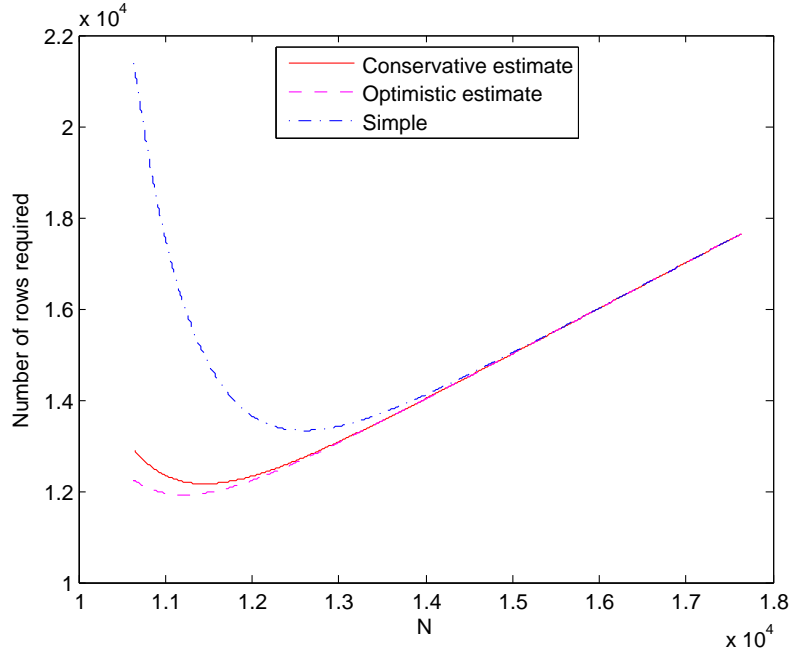


Figure 8: Size of the partial array vs. size of the complete CA. $t = 6$, $k = 56$, $v = 3$. Stein-Lovász-Johnson bound requires 17,403 rows, discrete Stein-Lovász-Johnson bound requires 13,021 rows. Simple estimate for the two stage algorithm is 13,328 rows, conservative estimate assuming $m = 2\gamma$ is 12,159 rows, and optimistic estimate assuming $m = \gamma$ is 11,919 rows. Even the conservative estimate beats the discrete Stein-Lovász-Johnson bound.

after the first stage, we encounter the same trade-off between time and storage, and final array size, as seen earlier for density. Hence we again parametrize by the expected number of uncovered interactions in the first stage.

We employ two different greedy algorithms to color the incompatibility graph. In method **Col** we first construct the incompatibility graph G after the first stage. Then we apply the commonly used *smallest last order* heuristic to order the vertices for greedy coloring: At each stage, find a vertex v_i of minimum degree in G_i , order the vertices of $G_i - v_i$, and then place v_i at the end. More precisely, we order the vertices of G as v_1, v_2, \dots, v_n , such that v_i is a vertex of minimum degree in G_i , where $G_i = G - \{v_{i+1}, \dots, v_n\}$. A graph is d -degenerate if all of its subgraphs have a vertex with degree at most d . When G is d -degenerate but not $(d-1)$ -degenerate, the *Coloring number* $\text{col}(G)$ is $d+1$. If we then greedily color the vertices with the first available color, at most $\text{col}(G)$ colors are used.

In method **Greedy** we employ an on-line, greedy approach that colors the interactions as they are discovered in the first stage. In this way, the incompatibility graph is never constructed. We instead maintain a set of rows. Some entries in rows are fixed to a specific value; others are flexible to take on any value. Whenever a new interaction is found to be uncovered in the first stage, we check if any of the rows is compatible with this interaction. If such a row is found then entries in the row are fixed so that the row now covers the interaction. If no such row exists, a new row with exactly t fixed entries corresponding to the interaction is added to the set of rows. This method is much faster than method **Col** in practice.

3.4 Using group action

Covering arrays that are invariant under the action of a permutation group on their symbols can be easier to construct and are often smaller [15]. Direct and computational constructions using group actions are explored in [9, 28]. Sarkar et al. [33] establish the asymptotically tightest known bounds on $\text{CAN}(t, k, v)$ using group actions. In this section we explore the implications of group actions on two-stage algorithms.

Let Γ be a permutation group on the set of symbols. The action of this group partitions the set of t -way interactions into orbits. We construct an array A such that for every orbit, at least one row covers an interaction from that orbit. Then we develop the rows of A over Γ to obtain a covering array that is invariant under the action of Γ . Effort then focuses on covering all the orbits of t -way interactions, instead of the individual interactions.

If Γ acts sharply transitively on the set of symbols (for example, if Γ is a cyclic group of order v) then the action of Γ partitions $\binom{k}{t}v^t$ interactions into $\binom{k}{t}v^{t-1}$ orbits of length v each. Following the lines of the proof of Theorem 2, there exists an $n \times k$ array with $n = \frac{\log \binom{k}{t} + (t-1) \log v + \log \log \left(\frac{v^{t-1}}{v^{t-1}-1} \right) + 1}{\log \left(\frac{v^{t-1}}{v^{t-1}-1} \right)}$ that covers at least one interaction from each orbit. Therefore,

$$\text{CAN}(t, k, v) \leq v \frac{\log \binom{k}{t} + (t-1) \log v + \log \log \left(\frac{v^{t-1}}{v^{t-1}-1} \right) + 1}{\log \left(\frac{v^{t-1}}{v^{t-1}-1} \right)}. \quad (1)$$

Similarly, we can employ a Frobenius group. When v is a prime power, the *Frobenius group* is the group of permutations of \mathbb{F}_v of the form $\{x \mapsto ax + b : a, b \in \mathbb{F}_v, a \neq 0\}$. The action of the Frobenius group partitions the set of t -tuples on v symbols into $\frac{v^{t-1}-1}{v-1}$ orbits of length $v(v-1)$ (full orbits) each and 1 orbit of length v (a short orbit). The short orbit consists of tuples of the form (x_1, \dots, x_t) where $x_1 = \dots = x_t$. Therefore, we can obtain a covering array by first constructing an array that covers all the full orbits, and then developing all the rows over the Frobenius group and adding v constant rows. Using the two stage strategy in conjunction with the Frobenius group action we obtain:

$$\text{CAN}(t, k, v) \leq v(v-1) \frac{\log \binom{k}{t} + \log \left(\frac{v^{t-1}-1}{v-1} \right) + \log \log \left(\frac{v^{t-1}}{v^{t-1}-v+1} \right) + 1}{\log \left(\frac{v^{t-1}}{v^{t-1}-v+1} \right)} + v. \quad (2)$$

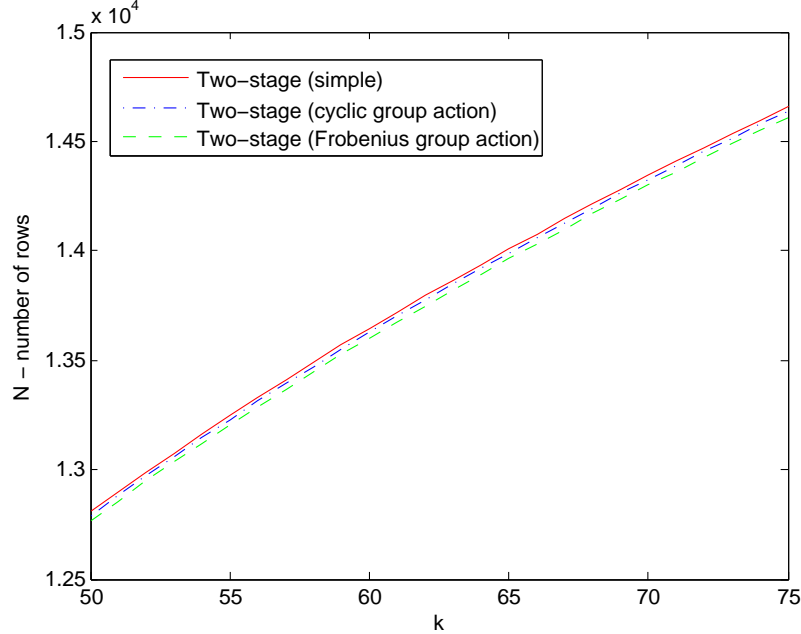


Figure 9: Comparison of the simple two-stage bound with the cyclic and the Frobenius two-stage bounds. $t = 6$, $v = 3$ and $50 \leq k \leq 75$. Group action reduces the required number of rows slightly.

Figure 9 compares the simple two-stage bound with the cyclic and Frobenius two-stage bounds. For $t = 6$, $v = 3$ and $12 \leq k \leq 100$, the cyclic bound requires 7-21 (on average 16) fewer rows than the simple bound. In the same range the Frobenius bound requires 17 – 51 (on average 40) fewer rows.

Group action can be applied in other methods for the second stage as well. Colbourn [15] incorporates group action into the density algorithm, allowing us to apply method **Den** in the second stage.

Greedy extends easily to use group action, as we do not construct an explicit incompatibility graph. Whenever we fix entries in a row to cover an uncovered orbit, we commit to a specific orbit representative.

However, applying group action to the incompatibility graph coloring for **Col** is more complicated. We need to modify the definition of the incompatibility graph for two reasons. First the vertices no longer represent uncovered interactions, but rather uncovered orbits of interaction. Secondly, and perhaps more importantly, pairwise compatibility between every two orbits in a set no longer implies mutual compatibility among all orbits in the set.

One approach is to form a vertex for each uncovered orbit, placing an edge between two when they share a column. Rather than the usual coloring, however, one asks for a partition of the vertex set into classes so that every class induces an acyclic subgraph. Problems of this type are *generalized graph coloring* problems [4]. Within each class of such a vertex partition, consistent representatives of each orbit can be selected to form a row; when a cycle is present, this may not be possible. Unfortunately, heuristics for solving these types of problems appear to be weak, so we adopt a second approach. As we build the incompatibility graph, we commit to specific orbit representatives. When a vertex for an uncovered orbit is added, we check its compatibility with the orbit representatives chosen for the orbits already handled with which it shares columns; we commit to an orbit representative and add edges to those with which it is now incompatible. Once completed, we have a (standard) coloring problem for the resulting graph.

Because group action can be applied using each of the methods for the two stages, we extend our naming to $\text{TS} \langle A, B; r, \Gamma \rangle$, where Γ can be Trivial (i.e. no group action), Cyclic, or Frobenius.

4 Computational results

Figure 5 indicates that even a simple two-stage bound can improve on best known covering array numbers. Therefore we investigate the actual performance of our two-stage algorithms for covering arrays of strength 5 and 6.

First we present results for $t = 6$, when $v \in \{3, 4, 5, 6\}$ and no group action is assumed. Table 1 shows the results for different v values. In each case we select the range of k values where the two-stage bound predicts smaller covering arrays than the previously known best ones, setting the maximum number of uncovered interactions as $\rho = 1/\log\left(\frac{v^t}{v^t-1}\right) \approx v^t$. For each value of k we construct a single partial array and then run the different second stage algorithms on it consecutively. In this way all the second stage algorithms cover the same set of uncovered interactions.

The column **tab** lists the best known $\text{CAN}(t, k, v)$ upper bounds from [13]. The column **bound** shows the upper bounds obtained from the two-stage bound (2). The columns **naïve**, **greedy**, **col** and **den** show results obtained from running the TS (Rand, Naive; ρ , Trivial), TS (Rand, Greedy; ρ , Trivial), TS (Rand, Col; ρ , Trivial) and TS (Rand, Den; ρ , Trivial) algorithms, respectively.

The naïve method always finds a covering array that is smaller than the two-stage bound. This happens because we repeat the first stage of Algorithm 4 until the array has fewer than v^t uncovered interactions. (If the first stage were not repeated, the algorithm still produce covering arrays that are not too far from the bound.) For $v = 3$ Greedy and Den have comparable performance. Method Col produces covering arrays that are smaller. However, for $v \in \{4, 5, 6\}$ Den and Col are competitive.

Table 2 shows the results obtained by the different second stage algorithms when the maximum number of uncovered interactions in the first stage is set to 2ρ and 3ρ respectively. When more interactions are covered in the second stage, we obtain smaller arrays as expected. However, the improvement in size does not approach 50%. There is no clear winner.

Next we investigate the covering arrays that are invariant under the action of a cyclic group. In Table 3 the column **bound** shows the upper bounds from Equation (1). The columns **naïve**, **greedy**, **col** and **den** show results obtained from running TS (Rand, Naive; ρ , Cyclic), TS (Rand, Greedy; ρ , Cyclic), TS (Rand, Col; ρ , Cyclic) and TS (Rand, Den; ρ , Cyclic), respectively.

Table 4 presents results for cyclic group action based algorithms when the number of maximum uncovered interactions in the first stage is set to 2ρ and 3ρ respectively.

For the Frobenius group action, we show results only for $v \in \{3, 5\}$ in Table 5. The column **bound** shows the upper bounds obtained from Equation (2).

Table 6 presents results for Frobenius group action algorithms when the number of maximum uncovered interactions in the first stage is 2ρ or 3ρ .

Next we present a handful of results when $t = 5$. In the cases examined, using the trivial group action is too time consuming to be practical. However, the cyclic or Frobenius cases are feasible. Tables 7 and 8 compare two stage algorithms when the number of uncovered interactions in the first stage is at most 2ρ .

In almost all cases there is no clear winner among the three second stage methods. Methods Den and Greedy are, however, substantially faster and use less memory than method Col; for practical purposes they would be preferred.

All code used in this experimentation is available from the github repository

<https://github.com/ksarkar/CoveringArray>

under an open source GPLv3 license.

5 Limited dependence and the Moser-Tardos algorithm

Here we explore a different randomized algorithm that produces smaller covering arrays than Algorithm 1. When $k > 2t$, there are interactions that share no column. The events of coverage of such interactions are independent. Moser et al. [29, 30] provide an efficient randomized construction method that exploits this

| k | tab | bound | naïve | greedy | col | den |
|----------------|---------|--------|--------|--------|--------|--------|
| $t = 6, v = 3$ | | | | | | |
| 53 | 13021 | 13076 | 13056 | 12421 | 12415 | 12423 |
| 54 | 14155 | 13162 | 13160 | 12510 | 12503 | 12512 |
| 55 | 17161 | 13246 | 13192 | 12590 | 12581 | 12591 |
| 56 | 19033 | 13329 | 13304 | 12671 | 12665 | 12674 |
| 57 | 20185 | 13410 | 13395 | 12752 | 12748 | 12757 |
| $t = 6, v = 4$ | | | | | | |
| 39 | 68314 | 65520 | 65452 | 61913 | 61862 | 61886 |
| 40 | 71386 | 66186 | 66125 | 62573 | 62826 | 62835 |
| 41 | 86554 | 66834 | 66740 | 63209 | 63160 | 63186 |
| 42 | 94042 | 67465 | 67408 | 63819 | 64077 | 64082 |
| 43 | 99994 | 68081 | 68064 | 64438 | 64935 | 64907 |
| 44 | 104794 | 68681 | 68556 | 65021 | 65739 | 65703 |
| $t = 6, v = 5$ | | | | | | |
| 31 | 233945 | 226700 | 226503 | 213244 | 212942 | 212940 |
| 32 | 258845 | 229950 | 229829 | 216444 | 217479 | 217326 |
| 33 | 281345 | 233080 | 232929 | 219514 | 219215 | 219241 |
| 34 | 293845 | 236120 | 235933 | 222516 | 222242 | 222244 |
| 35 | 306345 | 239050 | 238981 | 225410 | 226379 | 226270 |
| 36 | 356045 | 241900 | 241831 | 228205 | 230202 | 229942 |
| $t = 6, v = 6$ | | | | | | |
| 17 | 506713 | 486310 | 486302 | 449950 | 448922 | 447864 |
| 18 | 583823 | 505230 | 505197 | 468449 | 467206 | 466438 |
| 19 | 653756 | 522940 | 522596 | 485694 | 484434 | 483820 |
| 20 | 694048 | 539580 | 539532 | 502023 | 500788 | 500194 |
| 21 | 783784 | 555280 | 555254 | 517346 | 516083 | 515584 |
| 22 | 844834 | 570130 | 569934 | 531910 | 530728 | 530242 |
| 23 | 985702 | 584240 | 584194 | 545763 | 544547 | 548307 |
| 24 | 1035310 | 597660 | 597152 | 558898 | 557917 | 557316 |
| 25 | 1112436 | 610460 | 610389 | 571389 | 570316 | 569911 |
| 26 | 1146173 | 622700 | 622589 | 583473 | 582333 | 582028 |
| 27 | 1184697 | 634430 | 634139 | 594933 | 593857 | 593546 |

Table 1: Comparison of different TS $\langle \text{Rand}, -; \rho, \text{Trivial} \rangle$ algorithms.

| k | 2ρ | | | 3ρ | | |
|----------------|---------|--------|--------|---------|--------|--------|
| | greedy | col | den | greedy | col | den |
| $t = 6, v = 3$ | | | | | | |
| 53 | 11968 | 11958 | 11968 | 11716 | 11705 | 11708 |
| 54 | 12135 | 12126 | 12050 | 11804 | 11787 | 11790 |
| 55 | 12286 | 12129 | 12131 | 11877 | 11875 | 11872 |
| 56 | 12429 | 12204 | 12218 | 11961 | 12055 | 11950 |
| 57 | 12562 | 12290 | 12296 | 12044 | 12211 | 12034 |
| $t = 6, v = 4$ | | | | | | |
| 39 | 59433 | 59323 | 59326 | 58095 | 57951 | 57888 |
| 40 | 60090 | 60479 | 59976 | 58742 | 58583 | 58544 |
| 41 | 60715 | 61527 | 60615 | 59369 | 59867 | 59187 |
| 42 | 61330 | 62488 | 61242 | 59974 | 61000 | 59796 |
| 43 | 61936 | 61839 | 61836 | 60575 | 60407 | 60393 |
| 44 | 62530 | 62899 | 62428 | 61158 | 61004 | 60978 |
| $t = 6, v = 5$ | | | | | | |
| 31 | 204105 | 203500 | 203302 | 199230 | 198361 | 197889 |
| 32 | 207243 | 206659 | 206440 | 202342 | 201490 | 201068 |
| 33 | 210308 | 209716 | 209554 | 205386 | 204548 | 204107 |
| 34 | 213267 | 212675 | 212508 | 208285 | - | 207060 |
| 35 | 216082 | 215521 | 215389 | 211118 | - | 209936 |
| 36 | 218884 | 218314 | 218172 | 213872 | - | 212707 |
| $t = 6, v = 6$ | | | | | | |
| 17 | 425053 | - | 420333 | 412275 | - | 405093 |
| 18 | 443236 | - | 438754 | 430402 | - | 423493 |
| 19 | 460315 | - | 455941 | 447198 | - | 440532 |
| 20 | 476456 | - | 472198 | 463071 | - | 456725 |
| 21 | 491570 | - | 487501 | 478269 | - | 471946 |
| 22 | 505966 | - | 502009 | 492425 | - | 486306 |
| 23 | 519611 | - | 515774 | 505980 | - | 500038 |
| 24 | 532612 | - | 528868 | 518746 | - | 513047 |
| 25 | 544967 | - | 541353 | 531042 | - | 525536 |
| 26 | 556821 | - | 553377 | 542788 | - | 537418 |
| 27 | 568135 | - | 564827 | 554052 | - | 548781 |

Table 2: Comparison of TS $\langle \text{Rand}, -; 2\rho, \text{Trivial} \rangle$ and TS $\langle \text{Rand}, -; 3\rho, \text{Trivial} \rangle$ algorithms.

| k | tab | bound | naïve | greedy | col | den |
|----------------|---------|--------|--------|--------|--------|--------|
| $t = 6, v = 3$ | | | | | | |
| 53 | 13021 | 13059 | 13053 | 12405 | 12405 | 12411 |
| 54 | 14155 | 13145 | 13119 | 12489 | 12543 | 12546 |
| 55 | 17161 | 13229 | 13209 | 12573 | 12663 | 12663 |
| 56 | 19033 | 13312 | 13284 | 12660 | 12651 | 12663 |
| 57 | 20185 | 13393 | 13368 | 12744 | 12744 | 12750 |
| $t = 6, v = 4$ | | | | | | |
| k | tab | bound | naïve | greedy | col | den |
| 39 | 68314 | 65498 | 65452 | 61896 | 61860 | 61864 |
| 40 | 71386 | 66163 | 66080 | 62516 | 62820 | 62784 |
| 41 | 86554 | 66811 | 66740 | 63184 | 63144 | 63152 |
| 42 | 94042 | 67442 | 67408 | 63800 | 63780 | 63784 |
| 43 | 99994 | 68057 | 68032 | 64408 | 64692 | 64680 |
| 44 | 104794 | 68658 | 68556 | 64988 | 64964 | 64976 |
| $t = 6, v = 5$ | | | | | | |
| 31 | 226000 | 226680 | 226000 | 213165 | 212945 | 212890 |
| 32 | 244715 | 229920 | 229695 | 216440 | 217585 | 217270 |
| 33 | 263145 | 233050 | 233015 | 219450 | 221770 | 221290 |
| 34 | 235835 | 236090 | 235835 | 222450 | 222300 | 222210 |
| 35 | 238705 | 239020 | 238705 | 225330 | 225130 | 225120 |
| 36 | 256935 | 241870 | 241470 | 228140 | 229235 | 229020 |
| $t = 6, v = 6$ | | | | | | |
| 17 | 506713 | 486290 | 485616 | 449778 | 448530 | 447732 |
| 18 | 583823 | 505210 | 504546 | 468156 | 467232 | 466326 |
| 19 | 653756 | 522910 | 522258 | 485586 | 490488 | 488454 |
| 20 | 694048 | 539550 | 539280 | 501972 | 500880 | 500172 |
| 21 | 783784 | 555250 | 554082 | 517236 | 521730 | 519966 |
| 22 | 844834 | 570110 | 569706 | 531852 | 530832 | 530178 |
| 23 | 985702 | 584210 | 583716 | 545562 | 549660 | 548196 |
| 24 | 1035310 | 597630 | 597378 | 558888 | 557790 | 557280 |
| 25 | 1112436 | 610430 | 610026 | 571380 | 575010 | 573882 |
| 26 | 1146173 | 622670 | 622290 | 583320 | 582546 | 582030 |
| 27 | 1184697 | 624400 | 633294 | 594786 | 598620 | 597246 |

Table 3: Comparison of TS $\langle \text{Rand}, -; \rho, \text{Cyclic} \rangle$ algorithms.

| k | 2ρ | | | 3ρ | | |
|----------------|---------|--------|--------|---------|--------|--------|
| | greedy | col | den | greedy | col | den |
| $t = 6, v = 3$ | | | | | | |
| 53 | 11958 | 11955 | 11958 | 11700 | 11691 | 11694 |
| 54 | 12039 | 12027 | 12036 | 11790 | 11874 | 11868 |
| 55 | 12120 | 12183 | 12195 | 11862 | 12057 | 12027 |
| 56 | 12204 | 12342 | 12324 | 11949 | 11937 | 11943 |
| 57 | 12276 | 12474 | 12450 | 12027 | 12021 | 12024 |
| $t = 6, v = 4$ | | | | | | |
| 39 | 59412 | 59336 | 59304 | 58076 | 57976 | 57864 |
| 40 | 60040 | 59996 | 59964 | 58716 | 58616 | 58520 |
| 41 | 60700 | 61156 | 61032 | 59356 | 59252 | 59160 |
| 42 | 61320 | 62196 | 61976 | 59932 | 59840 | 59760 |
| 43 | 61908 | 63192 | 62852 | 60568 | 61124 | 60904 |
| 44 | 62512 | 64096 | 63672 | 61152 | 61048 | 60988 |
| $t = 6, v = 5$ | | | | | | |
| 31 | 204060 | 203650 | 203265 | 199180 | 198455 | 197870 |
| 32 | 207165 | 209110 | 208225 | 202255 | 204495 | 203250 |
| 33 | 207165 | 209865 | 209540 | 205380 | 204720 | 204080 |
| 34 | 213225 | 212830 | 212510 | 208225 | 207790 | 207025 |
| 35 | 216050 | 217795 | 217070 | 211080 | 213425 | 212040 |
| 36 | 218835 | 218480 | 218155 | 213770 | 213185 | 212695 |
| $t = 6, v = 6$ | | | | | | |
| 17 | 424842 | 422736 | 420252 | 411954 | 409158 | 405018 |
| 18 | 443118 | 440922 | 438762 | 430506 | 427638 | 423468 |
| 19 | 460014 | 457944 | 455994 | 447186 | 456468 | 449148 |
| 20 | 476328 | 474252 | 472158 | 463062 | 460164 | 456630 |
| 21 | 491514 | 489270 | 487500 | 478038 | 486180 | 479970 |
| 22 | 505884 | 503580 | 501852 | 492372 | 489336 | 486264 |
| 23 | 519498 | 517458 | 515718 | 505824 | 502806 | 500040 |
| 24 | 532368 | 530340 | 528828 | 518700 | 515754 | 512940 |
| 25 | 544842 | 542688 | 541332 | 530754 | 538056 | 532662 |
| 26 | 543684 | 543684 | 543684 | 542664 | 539922 | 537396 |
| 27 | 568050 | 566244 | 564756 | 553704 | 560820 | 555756 |

Table 4: Comparison of TS $\langle \text{Rand}, -; 2\rho, \text{Cyclic} \rangle$ and TS $\langle \text{Rand}, -; 3\rho, \text{Cyclic} \rangle$ algorithms.

| k | tab | bound | naïve | greedy | col | den |
|----------------|--------|--------|--------|--------|--------|--------|
| $t = 6, v = 3$ | | | | | | |
| 53 | 13021 | 13034 | 13029 | 12393 | 12387 | 12393 |
| 54 | 14155 | 13120 | 13071 | 12465 | 12513 | 12531 |
| 55 | 17161 | 13203 | 13179 | 12561 | 12549 | 12567 |
| 56 | 19033 | 13286 | 13245 | 12633 | 12627 | 12639 |
| 57 | 20185 | 13366 | 13365 | 12723 | 12717 | 12735 |
| $t = 6, v = 5$ | | | | | | |
| 31 | 233945 | 226570 | 226425 | 213025 | 212865 | 212865 |
| 32 | 258845 | 229820 | 229585 | 216225 | 216085 | 216065 |
| 33 | 281345 | 232950 | 232725 | 219285 | 219205 | 219145 |
| 34 | 293845 | 235980 | 234905 | 222265 | 223445 | 223265 |
| 35 | 306345 | 238920 | 238185 | 225205 | 227445 | 227065 |
| 36 | 356045 | 241760 | 241525 | 227925 | 231145 | 230645 |

Table 5: Comparison of TS (Rand, $-$; ρ , Frobenius) algorithms.

| k | 2ρ | | | 3ρ | | |
|----------------|---------|--------|--------|---------|--------|--------|
| | greedy | col | den | greedy | col | den |
| $t = 6, v = 3$ | | | | | | |
| 53 | 11931 | 11919 | 11931 | 11700 | 11691 | 11694 |
| 54 | 12021 | 12087 | 12087 | 11790 | 11874 | 11868 |
| 55 | 12105 | 12237 | 12231 | 11862 | 12057 | 12027 |
| 56 | 12171 | 12171 | 12183 | 11949 | 11937 | 11943 |
| 57 | 12255 | 12249 | 12255 | 12027 | 12021 | 12024 |
| 70 | 13167 | 13155 | 13179 | - | - | - |
| 75 | 13473 | 13473 | 13479 | - | - | - |
| 80 | 13773 | 13767 | 13779 | - | - | - |
| 85 | 14031 | 14025 | 14037 | - | - | - |
| 90 | 14289 | 14283 | 14301 | - | - | - |
| $t = 6, v = 5$ | | | | | | |
| 31 | 203785 | 203485 | 203225 | 198945 | 198445 | 197825 |
| 32 | 206965 | 208965 | 208065 | 201845 | 204505 | 203105 |
| 33 | 209985 | 209645 | 209405 | 205045 | 209845 | 207865 |
| 34 | 213005 | 214825 | 214145 | 208065 | 207545 | 206985 |
| 35 | 215765 | 215545 | 215265 | 210705 | 210365 | 209885 |
| 36 | 218605 | 218285 | 218025 | 213525 | 213105 | 212645 |
| 50 | 250625 | 250365 | 250325 | - | - | - |
| 55 | 259785 | 259625 | 259565 | - | - | - |
| 60 | 268185 | 268025 | 267945 | - | - | - |
| 65 | 275785 | 275665 | 275665 | - | - | - |

Table 6: Comparison of TS (Rand, $-$; 2ρ , Frobenius) and TS (Rand, $-$; 3ρ , Frobenius) algorithms.

| k | tab | greedy | col | den |
|-----|-------|--------|-------|-------|
| 67 | 59110 | 48325 | 48285 | 48305 |
| 68 | 60991 | 48565 | 48565 | 48585 |
| 69 | 60991 | 48765 | 49005 | 48985 |
| 70 | 60991 | 49005 | 48985 | 49025 |
| 71 | 60991 | 49245 | 49205 | 49245 |

Table 7: Comparison of TS $\langle \text{Rand}, -; 2\rho, \text{Frobenius} \rangle$ algorithms. $t = 5, v = 5$

| k | tab | greedy | col | den |
|-----|--------|--------|--------|--------|
| 49 | 122718 | 108210 | 108072 | 107988 |
| 50 | 125520 | 109014 | 108894 | 108822 |
| 51 | 128637 | 109734 | 110394 | 110166 |
| 52 | 135745 | 110556 | 110436 | 110364 |
| 53 | 137713 | 111306 | 111180 | 111120 |

Table 8: Comparison of TS $\langle \text{Rand}, -; 2\rho, \text{Cyclic} \rangle$ algorithms. $t = 5, v = 6$

limited dependence. Specializing their method to covering arrays, we obtain Algorithm 5. For the specified value of N in the algorithm it is guaranteed that the expected number of times the loop in line 3 of Algorithm 5 is repeated is linearly bounded in k (See Theorem 1.2 of [30]).

The upper bound on $\text{CAN}(t, k, v)$ guaranteed by Algorithm 5 is obtained by applying the Lovász local lemma (LLL).

Lemma 4. (*Lovász local lemma; symmetric case*) (see [2]) *Let A_1, A_2, \dots, A_n events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all other events A_j except for at most d , and that $\Pr[A_i] \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) \leq 1$, then $\Pr[\cap_{i=1}^n \bar{A}_i] > 0$.*

The symmetric version of Lovász local lemma provides an upper bound on the probability of a “bad” event in terms of the maximum degree of a bad event in a dependence graph, so that the probability that all the bad events are avoided is non zero. Godbole, Skipper, and Sunley [18] apply Lemma 4 essentially to obtain the bound on $\text{CAN}(t, k, v)$ in line 1 of Algorithm 5.

Theorem 5. [18] *Let t, v and $k \geq 2t$ be integers with $t, v \geq 2$. Then*

$$\text{CAN}(t, k, v) \leq \frac{\log \left\{ \binom{k}{t} - \binom{k-t}{t} \right\} + t \log v + 1}{\log \left(\frac{v^t}{v^t - 1} \right)}$$

The bound on the size of covering arrays obtained from Theorem 5 is asymptotically tighter than the one obtained from Theorem 1. Figure 10 compares the bounds for $t = 6$ and $v = 3$.

The original proof of LLL is essentially non-constructive and does not immediately lead to a polynomial time construction algorithm for covering arrays satisfying the bound of Theorem 5. Indeed no previous construction algorithms appear to be based on it. However the Moser-Tardos method of Algorithm 5 does provide a construction algorithm running in expected polynomial time. For sufficiently large values of k Algorithm 5 produces smaller covering arrays than the Algorithm 1.

But the question remains: Does Algorithm 5 produce smaller covering arrays than the currently best known results within the range that it can be effectively computed? Perhaps surprisingly, we show that the answer is affirmative. In Algorithm 5 we do not need to store the coverage information of individual interactions in memory because each time an uncovered interaction is encountered we re-sample the columns involved in that interaction and start the check afresh (checking the coverage in interactions in the same order each time). Consequently, Algorithm 5 can be applied for larger values of k than the density algorithm.

Algorithm 5: Moser-Tardos type algorithm for covering array construction.

Input: t : strength of the covering array, k : number of factors, v : number of levels for each factor
Output: A : a $\text{CA}(N; t, k, v)$

```

1 Let  $N := \frac{\log\left\{\binom{k}{t} - \binom{k-t}{t}\right\} + t \cdot \log v + 1}{\log\left(\frac{v^t}{v^t - 1}\right)}$ ;
2 Construct an  $N \times k$  array  $A$  where each entry is chosen independently and uniformly at random from a  $v$ -ary alphabet;
3 repeat
4   Set  $\text{covered} := \text{true}$ ;
5   for each interaction  $\iota \in \mathcal{I}_{t,k,v}$  do
6     if  $\iota$  is not covered in  $A$  then
7       Set  $\text{covered} := \text{false}$ ;
8       Set  $\text{missing-interaction} := \iota$ ;
9       break;
10    end
11  end
12  if  $\text{covered} = \text{false}$  then
13    Choose all the entries in the  $t$  columns involved in  $\text{missing-interaction}$  independently and uniformly at random from the  $v$ -ary alphabet;
14  end
15 until  $\text{covered} = \text{true}$ ;
16 Output  $A$ ;
```

Smaller covering arrays can be obtained by exploiting a group action using LLL, as shown in [33]. Table 9 shows the sizes of the covering arrays constructed by a variant of Algorithm 5 that employs cyclic and Frobenius group actions. While this single stage algorithm produces smaller arrays than the currently best known [13], these are already superseded by the two-stage based algorithms.

| k | tab | MT | k | tab | MT | k | tab | MT |
|----|-------|-------|----|--------|--------|----|---------|---------|
| 56 | 19033 | 16281 | 44 | 411373 | 358125 | 25 | 1006326 | 1020630 |
| 57 | 20185 | 16353 | 45 | 417581 | 360125 | 26 | 1040063 | 1032030 |
| 58 | 23299 | 16425 | 46 | 417581 | 362065 | 27 | 1082766 | 1042902 |
| 59 | 23563 | 16491 | 47 | 423523 | 363965 | 28 | 1105985 | 1053306 |
| 60 | 23563 | 16557 | 48 | 423523 | 365805 | 29 | 1149037 | 1063272 |

(a) Frobenius. $t = 6, v = 3$ (b) Frobenius. $t = 6, v = 5$ (c) Cyclic. $t = 6, v = 6$

Table 9: Comparison of covering array size from Algorithm 5 (MT) with the best known results [13] (tab).

5.1 Moser-Tardos type algorithm for the first stage

The linearity of expectation arguments used in the SLJ bounds permit one to consider situations in which a few of the “bad” events are allowed to occur, a fact that we exploited in the first stage of the algorithms thus far. However, the Lovász local lemma does not address this situation directly. The conditional Lovász local lemma (LLL) distribution, introduced in [19], is a very useful tool.

Lemma 6. (Conditional LLL distribution; symmetric case) (see [2, 33]) Let $\mathcal{A} = \{A_1, A_2, \dots, A_l\}$ be a set of l events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all other events A_j except for at most d , and that $\Pr[A_i] \leq p$ for all $1 \leq i \leq l$. Also suppose that $ep(d+1) \leq 1$ (Therefore, by LLL (Lemma 4) $\Pr[\cap_{i=1}^l \bar{A}_i] > 0$). Let $B \notin \mathcal{A}$ be another event in the same probability space

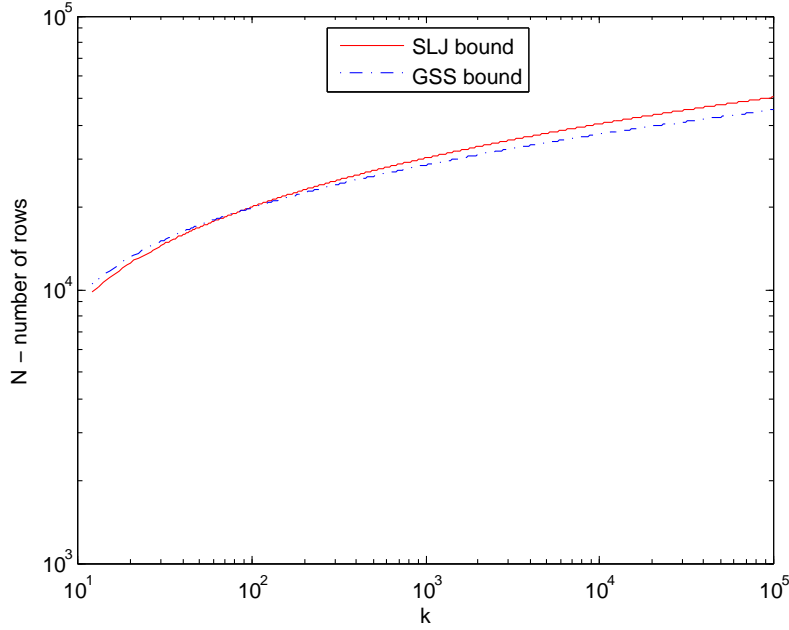


Figure 10: Comparison of SLJ (Theorem 1) and GSS (Theorem 5) bounds for $t = 6$ and $v = 3$. The graph is plotted in log-log scale to highlight the asymptotic difference between the two bounds.

with $\Pr[B] \leq q$, such that B is also mutually independent of a set of all other events $A_j \in \mathcal{A}$ except for at most d . Then $\Pr[B | \cap_{i=1}^l \bar{A}_i] \leq eq$.

We apply the conditional LLL distribution to obtain an upper bound on the size of partial array that leaves at most $\log\left(\frac{v^t}{v^t-1}\right) \approx v^t$ interactions uncovered. For a positive integer k , let $I = \{j_1, \dots, j_\rho\} \subseteq [k]$ where $j_1 < \dots < j_\rho$. Let A be an $n \times k$ array where each entry is from the set $[v]$. Let A_I denote the $n \times \rho$ array in which $A_I(i, \ell) = A(i, j_\ell)$ for $1 \leq i \leq n$ and $1 \leq \ell \leq \rho$; A_I is the projection of A onto the columns in I .

Let $M \subseteq [v]^t$ be a set of m t -tuples of symbols, and $C \in \binom{[k]}{t}$ be a set of t columns. Suppose the entries in the array A are chosen independently from $[v]$ with uniform probability. Let B_C denote the event that at least one of the tuples in M is not covered in A_C . There are $\eta = \binom{k}{t}$ such events, and for all of them $\Pr[B_C] \leq m \left(1 - \frac{1}{v^t}\right)^n$. Moreover, when $k \geq 2t$, each of the events is mutually independent of all other events except for at most $\rho = \binom{k}{t} - \binom{k-t}{t} - 1 < t \binom{k}{t-1}$. Therefore, by the Lovász local lemma, when $e\rho m \left(1 - \frac{1}{v^t}\right)^n \leq 1$, none of the events B_C occur. Solving for n , when

$$n \geq \frac{\log(epm)}{\log\left(\frac{v^t}{v^t-1}\right)} \quad (3)$$

there exists an $n \times k$ array A over $[v]$ such that for all $C \in \binom{[k]}{t}$, A_C covers all the m tuples in M . In fact we can use a Moser-Tardos type algorithm to construct such an array.

Let ι be an interaction whose t -tuple of symbols is not in M . Then the probability that ι is not covered in an $n \times k$ array is at most $\left(1 - \frac{1}{v^t}\right)^n$ when each entry of the array is chosen independently from $[v]$ with uniform probability. Therefore, by the conditional LLL distribution the probability that ι is not covered in the array A where for all $C \in \binom{[k]}{t}$, A_C covers all the m tuples in M is at most $e \left(1 - \frac{1}{v^t}\right)^n$. Moreover, there are $\eta(v^t - m)$ such interactions ι . By the linearity of expectation, the expected number of uncovered

interactions in A is less than v^t when $\eta(v^t - m)e(1 - \frac{1}{v^t})^n \leq v^t$. Solving for n , we obtain

$$n \geq \frac{\log \left\{ \eta e \left(1 - \frac{m}{v^t} \right) \right\}}{\log \left(\frac{v^t}{v^t - 1} \right)}. \quad (4)$$

Therefore, there exists an $n \times k$ array with $n = \max \left\{ \frac{\log(e\rho m)}{\log(\frac{v^t}{v^t-1})}, \frac{\log\{\eta e(1 - \frac{m}{v^t})\}}{\log(\frac{v^t}{v^t-1})} \right\}$ that has at most v^t uncovered interactions. To compute n explicitly, we must choose m . We can select a value of m to minimize n graphically for given values of t , k and v . For example, Figure 11 plots Equations 3 and 4 against m for $t = 3$, $k = 350$, $v = 3$, and finds the minimum value of n .

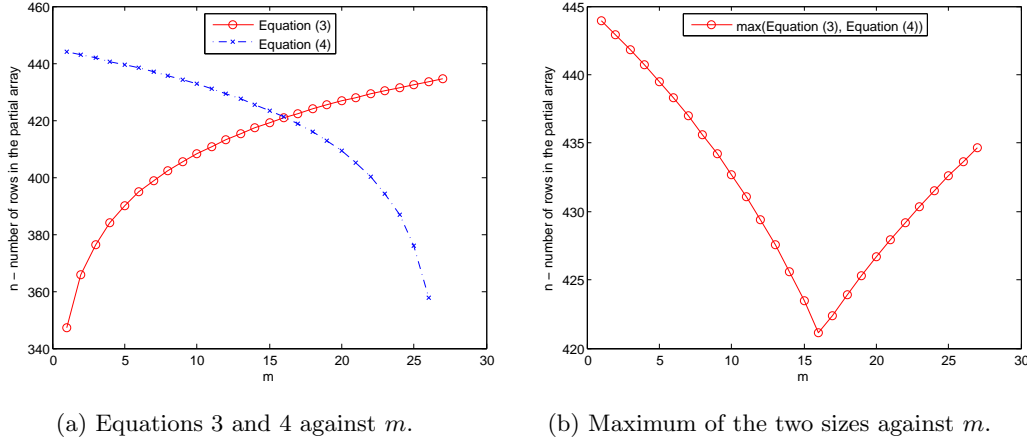


Figure 11: The minimum is at $n = 422$, when $m = 16$. $t = 3$, $k = 350$, $v = 3$

We compare the size of the partial array from the naïve two-stage method (Algorithm 4) with the size obtained by the graphical methods in Figure 12. The Lovász local lemma based method is asymptotically better than the simple randomized method. However, except for the small values of t and v , in the range of k values relevant for practical applications the simple randomized algorithm requires fewer rows than the Lovász local lemma based method.

5.2 Lovász local lemma based two-stage bound

We can apply the techniques from Section 5.1 to obtain a two-stage bound similar to Theorem 2 using the Lovász local lemma and conditional LLL distribution. First we extend a result from [33].

Theorem 7. *Let t, k, v be integers with $k \geq t \geq 2$, $v \geq 2$ and let $\eta = \binom{k}{t}$, and $\rho = \binom{k}{t} - \binom{k-t}{t}$. If $\frac{\eta v^t \log(\frac{v^t}{v^t-1})}{\rho} \leq v^t$ Then*

$$\text{CAN}(t, k, v) \leq \frac{\log \binom{k}{t} + t \log v + \log \log \left(\frac{v^t}{v^t-1} \right) + 2}{\log \left(\frac{v^t}{v^t-1} \right)} - \frac{\eta}{\rho}.$$

Proof. Let $M \subseteq [v]^t$ be a set of m t -tuples of symbols. Following the arguments of Section 5.1, when $n \geq \frac{\log(e\rho m)}{\log(\frac{v^t}{v^t-1})}$ there exists an $n \times k$ array A over $[v]$ such that for all $C \in \binom{[k]}{t}$, A_C covers all m tuples in M .

At most $\eta(v^t - m)$ interactions are uncovered in such an array. Using the conditional LLL distribution, the probability that one such interaction is not covered in A is at most $e(1 - \frac{1}{v^t})^n$. Therefore, by the

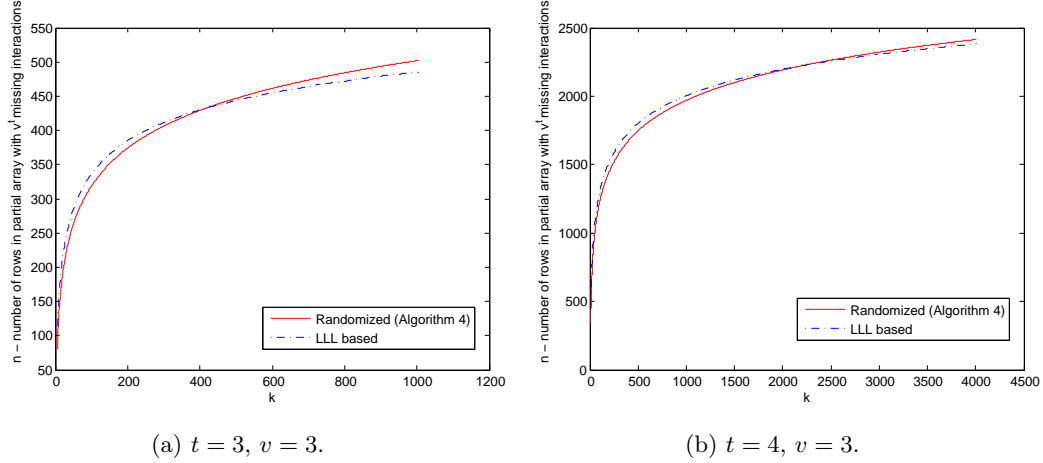


Figure 12: Comparison of the size of the partial array constructed in the first stage. Compares the size of the partial array specified in Algorithm 4 in Section 3.1, and the size derived in Section 5.1.

linearity of expectation, we can find one such array A that leaves at most $e\eta(v^t - m)(1 - \frac{1}{v^t})^n = \frac{\eta}{\rho}(\frac{v^t}{m} - 1)$ interactions uncovered. Adding one row per uncovered interactions to A , we obtain a covering array with at most N rows, where

$$N = \frac{\log(e\rho m)}{\log\left(\frac{v^t}{v^t-1}\right)} + \frac{\eta}{\rho}\left(\frac{v^t}{m} - 1\right)$$

The value of N is minimized when $m = \frac{\eta v^t \log\left(\frac{v^t}{v^t-1}\right)}{\rho}$. Because $m \leq v^t$, we obtain the desired bound. \square

When $m = v^t$ this recaptures the bound of Theorem 5.

Figure 13 compares the LLL based two-stage bound from Theorem 7 to the standard two-stage bound from Theorem 2, the Godbole et al. bound in Theorem 5, and the SLJ bound in Theorem 1. Although the LLL based two-stage bound is tighter than the LLL based Godbole et al. bound, even for quite large values of k the standard two-stage bound is tighter than the LLL based two-stage bound. In practical terms, this specific LLL based two-stage method does not look very promising, unless the parameters are quite large.

6 Conclusion and open problems

Many concrete algorithms within a two-stage framework for covering array construction have been introduced and evaluated. The two-stage approach extends the range of parameters for which competitive covering arrays can be constructed, each meeting an *a priori* guarantee on its size. Indeed as a consequence a number of best known covering arrays have been improved upon. Although each of the methods proposed has useful features, our experimental evaluation suggests that $\text{TS}(\text{Rand}, \text{Greedy}; 2\rho, \Gamma)$ and $\text{TS}(\text{Rand}, \text{Den}; 2\rho, \Gamma)$ with $\Gamma \in \{\text{Cyclic}, \text{Frobenius}\}$ realize a good trade-off between running time and size of the constructed covering array.

Improvements in the bounds, or in the algorithms that realize them, are certainly of interest. We mention some specific directions. Establishing tighter bounds on the coloring based methods of Section 3.3 is a challenging problem. Either better estimates of the chromatic number of the incompatibility graph after a random first stage, or a first stage designed to limit the chromatic number, could lead to improvements in the bounds.

In Section 5.1 and 5.2 we explored using a Moser-Tardos type algorithm for the first stage. Although this is useful for asymptotic bounds, practical improvements appear to be limited. Perhaps a different approach of

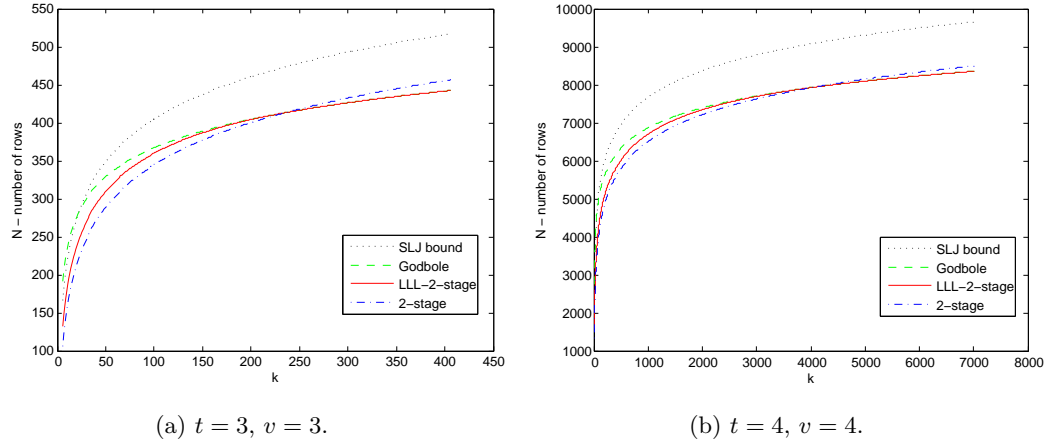


Figure 13: Comparison among the LLL based two-stage bound from Theorem 7, the standard two-stage bound from Theorem 2, the Godbole et al. bound in Theorem 5, and the SLJ bound in Theorem 1.

reducing the number of bad events to be avoided explicitly by the algorithm may lead to a better algorithm. A potential approach may look like following: “Bad” events would denote non-coverage of an interaction over a t -set of columns. We would select a set of column t -sets such that the dependency graph of the corresponding bad events have a bounded maximum degree (less than the original dependency graph). We would devise a Moser-Tardos type algorithm for covering all the interactions on our chosen column t -sets, and then apply the conditional LLL distribution to obtain an upper bound on the number of uncovered interactions. However, the difficulty lies in the fact that “all vertices have degree $\leq \rho$ ” is a non-trivial, “hereditary” property for induced subgraphs, and for such properties finding a maximum induced subgraph with the property is an NP-hard optimization problem [17]. There is still hope for a randomized or “nibble” like strategy that may find a reasonably good induced subgraph with a bounded maximum degree. Further exploration of this idea seems to be a promising research avenue.

In general, one could consider more than two stages. Establishing the benefit (or not) of having more than two stages is also an interesting open problem. Finally, the application of the methods developed to mixed covering arrays appears to provide useful techniques for higher strengths; this merits further study as well.

Acknowledgments

The research was supported in part by the National Science Foundation under Grant No. 1421058.

References

- [1] A. N. Aldaco, C. J. Colbourn, and V. R. Syrotiuk. Locating arrays: A new experimental design for screening complex engineered systems. *SIGOPS Oper. Syst. Rev.*, 49(1):31–40, Jan. 2015.
- [2] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.
- [3] Android. Android configuration class, 2016. <http://developer.android.com/reference/android/content/res/Configuration.html>.
- [4] J. I. Brown. The complexity of generalized graph colorings. *Discrete Appl. Math.*, 69(3):257–270, 1996.

- [5] R. C. Bryce and C. J. Colbourn. The density algorithm for pairwise interaction testing. *Software Testing, Verification, and Reliability*, 17:159–182, 2007.
- [6] R. C. Bryce and C. J. Colbourn. A density-based greedy algorithm for higher strength covering arrays. *Software Testing, Verification, and Reliability*, 19:37–53, 2009.
- [7] R. C. Bryce and C. J. Colbourn. Expected time to detection of interaction faults. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 86:87–110, 2013.
- [8] J. N. Cawse. Experimental design for combinatorial and high throughput materials development. *GE Global Research Technical Report*, 29:769–781, 2002.
- [9] M. A. Chateauneuf, C. J. Colbourn, and D. L. Kreher. Covering arrays of strength 3. *Des. Codes Crypt.*, 16:235–242, 1999.
- [10] D. M. Cohen, S. R. Dalal, M. L. Fredman, and G. C. Patton. The AETG system: An approach to testing based on combinatorial design. *IEEE Transactions on Software Engineering*, 23:437–44, 1997.
- [11] M. B. Cohen. *Designing test suites for software interaction testing*. PhD thesis, The University of Auckland, Department of Computer Science, 2004.
- [12] C. J. Colbourn. Combinatorial aspects of covering arrays. *Le Matematiche (Catania)*, 58:121–167, 2004.
- [13] C. J. Colbourn. Covering array tables, 2005-2015. <http://www.public.asu.edu/~ccolbou/src/tabby>.
- [14] C. J. Colbourn. Covering arrays and hash families. In *Information Security and Related Combinatorics*, NATO Peace and Information Security, pages 99–136. IOS Press, 2011.
- [15] C. J. Colbourn. Conditional expectation algorithms for covering arrays. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 90:97–115, 2014.
- [16] R. Diestel. *Graph Theory*. Graduate Texts in Mathematics. Springer, fourth edition, 2010.
- [17] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.
- [18] A. P. Godbole, D. E. Skipper, and R. A. Sunley. t -covering arrays: upper bounds and Poisson approximations. *Combinatorics, Probability and Computing*, 5:105–118, 1996.
- [19] B. Haeupler, B. Saha, and A. Srinivasan. New constructive aspects of the Lovász local lemma. *J. ACM*, 58(6):Art. 28, 28, 2011.
- [20] A. S. Hedayat, N. J. A. Sloane, and J. Stufken. *Orthogonal Arrays*. Springer-Verlag, New York, 1999.
- [21] D. S. Johnson. Approximation algorithms for combinatorial problems. *J. Comput. System Sci.*, 9:256–278, 1974.
- [22] G. O. H. Katona. Two applications (for search theory and truth functions) of Sperner type theorems. *Periodica Math.*, 3:19–26, 1973.
- [23] D. Kleitman and J. Spencer. Families of k -independent sets. *Discrete Math.*, 6:255–262, 1973.
- [24] D. R. Kuhn, R. Kacker, and Y. Lei. *Introduction to Combinatorial Testing*. CRC Press, 2013.
- [25] D. R. Kuhn and M. Reilly. An investigation of the applicability of design of experiments to software testing. In *Proc. 27th Annual NASA Goddard/IEEE Software Engineering Workshop*, pages 91–95, Los Alamitos, CA, 2002. IEEE.

- [26] D. R. Kuhn, D. R. Wallace, and A. M. Gallo. Software fault interactions and implications for software testing. *IEEE Trans. Software Engineering*, 30:418–421, 2004.
- [27] L. Lovász. On the ratio of optimal integral and fractional covers. *Discrete Math.*, 13(4):383–390, 1975.
- [28] K. Meagher and B. Stevens. Group construction of covering arrays. *J. Combin. Des.*, 13:70–77, 2005.
- [29] R. A. Moser. A constructive proof of the Lovász local lemma. In *STOC’09—Proceedings of the 2009 ACM International Symposium on Theory of Computing*, pages 343–350. ACM, New York, 2009.
- [30] R. A. Moser and G. Tardos. A constructive proof of the general Lovász local lemma. *J. ACM*, 57(2):Art. 11, 15, 2010.
- [31] C. Nie and H. Leung. A survey of combinatorial testing. *ACM Computing Surveys*, 43(2):#11, 2011.
- [32] A. H. Ronneseth and C. J. Colbourn. Merging covering arrays and compressing multiple sequence alignments. *Discrete Appl. Math.*, 157:2177–2190, 2009.
- [33] K. Sarkar and C. J. Colbourn. Upper bounds on the size of covering arrays. *ArXiv e-prints*.
- [34] G. Seroussi and N. H. Bshouty. Vector sets for exhaustive testing of logic circuits. *IEEE Trans. Inform. Theory*, 34:513–522, 1988.
- [35] S. K. Stein. Two combinatorial covering theorems. *J. Combinatorial Theory Ser. A*, 16:391–397, 1974.
- [36] J. Torres-Jimenez, H. Avila-George, and I. Izquierdo-Marquez. A two-stage algorithm for combinatorial testing. *Optimization Letters*, pages 1–13, 2016.